Fine-Grained Analysis of Problems on Curves

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Fréchet Distance between two curves

Fréchet distance [Alt and Godau, 1995]

Minimize the maximal distance between curves $P$ and $Q$

Upper bound [Agarwal et al., 2014]

Running time $O\left(\frac{mn \log \log n}{\log n}\right)$ for the discrete Fréchet distance

Lower bound [Bringmann, 2014]

Discrete Fréchet distance cannot be computed in $O(n^{2-\varepsilon})$ for any $\varepsilon > 0$ unless the strong exponential time hypothesis fails
How can we capture distances on a tuple of points?

An alignment $C = \langle C_1, \ldots, C_m \rangle$ of the curves $A_1, A_2, A_3$

$C_1 = (0, 0, \ldots, 0)$

$C_m = (n_1, n_2, \ldots, n_k)$

$C_{i+1}[h] = C_i[h]$ or $C_{i+1}[h] = C_i[h] + 1$
Fréchet Distance between $k$ curves

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Discrete Fréchet distance: minimize distance over all coupled distances $d_C$

Upper bound [Dumitrescu and Rote, 2004]

Running time $O(n^k)$ for $k$ polygonal curves
min-# Simplification problem:
• Given a polygonal curve $P$ and an $\varepsilon > 0$ as an error threshold
• Objective: minimize the number of vertices in a simplification $S$
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min-# Simplification problem:
• Given a polygonal curve $P$ and an $\varepsilon > 0$ as an error threshold
Upper bound [Chan and Chin, 1996]
A min-# simplification can be computed in $O(n^2)$ time in $\mathbb{R}^2$

Higher dimensions [Barequet et al., 2002]
For the $L_1$ or $L_\infty$ metric, a min-# simplification can be computed in $O(n^2)$ time
1. Result

- No $O(n^{k-\varepsilon})$ time algorithm for the discrete Fréchet distance on $k$ curves for any $\varepsilon > 0$ unless SETH fails.
Lower Bounds

1. Result
   - No \( O(n^{k-\varepsilon}) \) time algorithm for the discrete Fréchet distance on \( k \) curves for any \( \varepsilon > 0 \) unless SETH fails

2. Result
   - No \( O(n^{2-\varepsilon}) \) time algorithm for curve simplification with \( d = \Omega(\log n) \) dimensions for any \( \varepsilon > 0 \) unless SETH fails
Orthogonal Vectors

Proof idea

- Transform $k$ Orthogonal Vectors to curves
  1. Gadgets for coordinates
  2. Synchronized walk of the composite curves
- Fréchet distance $d_F(\cdot) \leq 1$ iff. vectors are orthogonal
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\[
\begin{pmatrix}
\alpha^1 \\
\alpha^2 \\
\vdots \\
\alpha^k
\end{pmatrix}
= 
\begin{pmatrix}
0 & 1 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 1 & \ldots & 1 & 1
\end{pmatrix}
\]

$\alpha^1, \alpha^2, \ldots, \alpha^k$ are non-orthogonal
Orthogonal Vectors

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\begin{pmatrix}
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\end{pmatrix}
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\end{pmatrix}
\]

\(\alpha^1, \alpha^2, \ldots, \alpha^k\) are orthogonal
Orthogonal Vectors

\( k \) Orthogonal Vectors

- Given \( k \ \{0, 1\}^d \) vectors \( \alpha^1, \alpha^2, \ldots, \alpha^k \)
- Do they satisfy

\[
\sum_{h=1}^{d} \prod_{t \in [k]} \alpha^t[h] = 0?
\]

\[
\begin{pmatrix}
\alpha^1 \\
\alpha^2 \\
\vdots \\
\alpha^k \\
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
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\end{pmatrix}
\]

\( \alpha^1, \alpha^2, \ldots, \alpha^k \) are orthogonal

Strong Exponential Time Hypothesis (SETH)

- For every \( \varepsilon \) there is a \( k \) such that
- SAT on \( k \)-CNF cannot be solved in subexponential time
Encoding coordinates from $k$-Orthogonal Vectors $A_1, A_2, \ldots, A_{k-1}$ and $B$ by curves

$-0.5 - \delta \quad -0.5 - \delta \quad 0.5 + \delta$

$CG_i(0) \quad CG_i(1) \quad CG_B(0) \quad CG_B(1)$
Example for Coordinate Gadgets

\[ \alpha_1^1 = 0 \]
\[ \alpha_2^1 = 1 \]
\[ \beta_1 = 0 \]

\[ -0.5 - \delta \]
\[ 0.5 + \delta \]
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\[ \beta_1 = 0 \]

\[ -0.5 - \delta \]

\[ CG_1(0) \]

\[ CG_2(1) \]

\[ CG_B(0) \]

\[ 0.5 + \delta \]

\[ d_F(CG_1(0), CG_2(1), CG_B(0)) \leq 1 \]
Proof idea

\[ s, t_A, t_B \in CG_* \]

\[ v_A, v_B \]
Proof idea

1. Simultaneous walk of curves $A_1, ... A_{k-1}$ from $s$ to $v_A$
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2. Curve $B$ walks to $v_B$ over $s, v_A$
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3. Synchronized traversal of all coordinate gadgets $CG_*$
Proof idea

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4. Curves $A_1, \ldots, A_{k-1}$ walk to $t_A$ simultaneously
Proof idea

1. Simultaneous walk of curves $A_1, \ldots, A_{k-1}$ from $s$ to $v_A$
2. Curve $B$ walks to $v_B$ over $s, v_A$
3. Synchronized traversal of all coordinate gadgets $CG^*$
4. Curves $A_1, \ldots, A_{k-1}$ walk to $t_A$ simultaneously
5. First $B$ terminates at $s$, then $A_1, \ldots, A_{k-1}$
Proof construction

- Reduction from 2 Orthogonal Vectors $\alpha, \beta$ in $d$ dimensions
- to Curve Simplification in $d + 1$ dimensions
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- to Curve Simplification in $d + 1$ dimensions

- $\alpha = 0, \beta = 0$
- $\alpha = 0, \beta = 1$
- $\alpha = 1, \beta = 1$
Proof construction

- Reduction from 2 Orthogonal Vectors $\alpha, \beta$ in $d$ dimensions
- to Curve Simplification in $d + 1$ dimensions

Proof idea

- Checkpoints $q$ are dropped within the simplification iff. $\alpha, \beta$ are orthogonal
- Simplification consists of 4 vertices iff. $\alpha, \beta$ are orthogonal
- Simplification consists of at least 5 vertices iff. $\alpha, \beta$ are non-orthogonal
Conclusion

1. Result
   - No $O(n^{k-\varepsilon})$ time algorithm for the discrete Fréchet distance on $k$ curves for any $\varepsilon > 0$ unless SETH fails

2. Result
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Open problems

- Lower bound for simplification in $\mathbb{R}^2$ or $\mathbb{R}^3$
- Upper bound for simplification in $\mathbb{R}^2$ or $\mathbb{R}^3$
- Lower bound for Fréchet distance on $k$ curves with dimension $d = 1$

Thank you for your attention.
Simplification: Orthogonal Case

$L_\infty: \varepsilon = 1, \delta = 2, \delta' = 0.5$

$\alpha = \langle 0, 1, 0 \rangle$
$\beta = \langle 1, 0, 0 \rangle$
$s = e = (0, 0)$

$\hat{\alpha} = \langle (0, -2), (1, -2), (0, -2) \rangle$
$\hat{\beta} = \langle (1, 2), (0, 2), (0, 2) \rangle$
$q = (-0.5, 0)$
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Simplification contains 4 vertices iff. $\alpha$ and $\beta$ are orthogonal
Simplification: Non-orthogonal Case

$L_{\infty}$: $\epsilon = 1$, $\delta = 2$, $\delta' = 0.5$

$\alpha = \langle 1, 0, 1 \rangle$

$\beta = \langle 0, 0, 1 \rangle$

$s = e = (0, 0)$

$\hat{\alpha} = \langle 1, -2, 0, -2, 1, -2 \rangle$

$\hat{\beta} = \langle 0, 2, 0, 2, 1, 2 \rangle$

$q = (-0.5, 0)$
Simplification: Non-orthogonal Case

$L_\infty : \varepsilon = 1, \delta = 2, \delta' = 0.5$

$\alpha = \langle 1, 0, 1 \rangle$
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$q = (-0.5, 0)$

Simplification contains at least 5 vertices iff. $\alpha$ and $\beta$ are non-orthogonal.