Low-Crossing Spanning Trees

An Alternative Proof and Experiments

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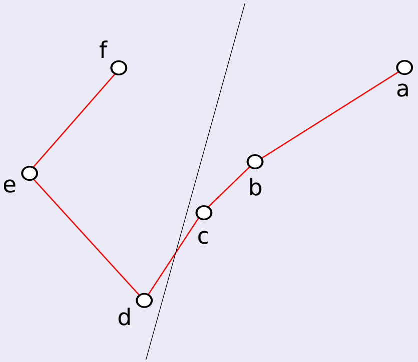
Where innovation starts

Spanning Trees with Low Crossing Number

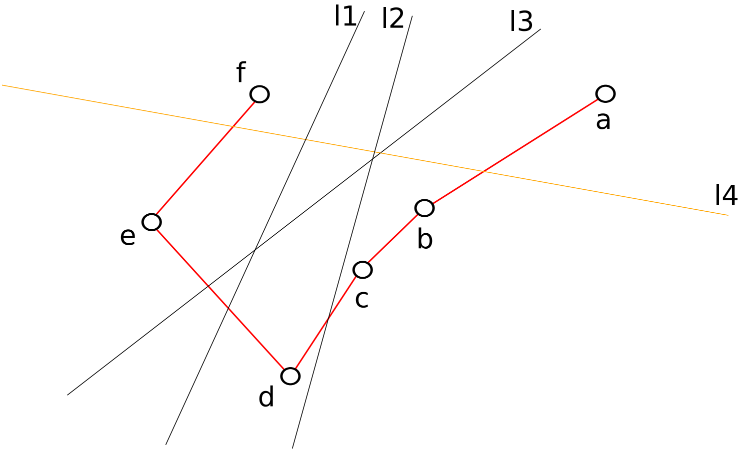
Our Results

1. Simple proof on low-crossing spanning trees
2. A new heuristic to compute those trees
3. Experimental results on
   - artificial data
   - real TSP instances from TSPLIB

What is a crossing?

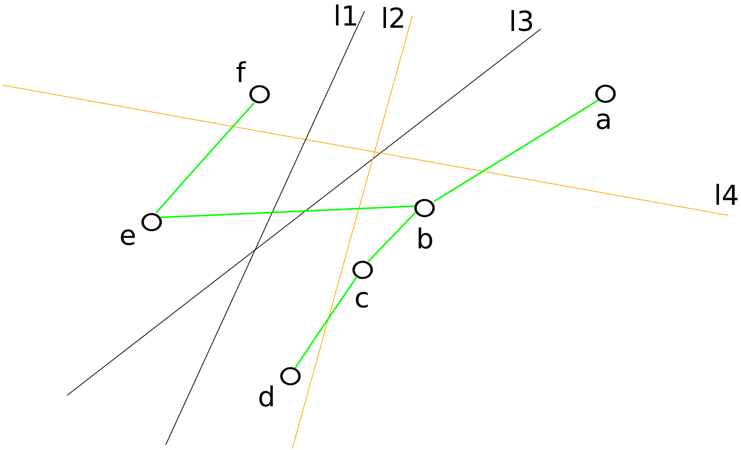


What is a Low-crossing Spanning Tree?



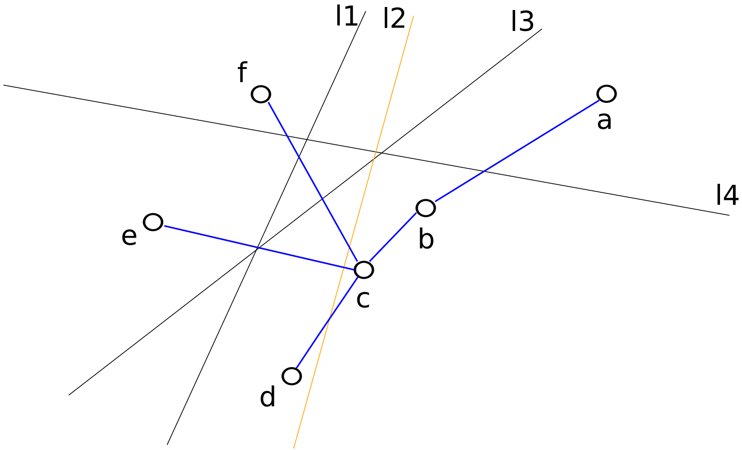
A spanning tree F with crossing number 2

Is there only one optimum?



Another spanning tree F with crossing number 2

Is this an optimum?



Non optimal spanning tree F with crossing number 3

Preliminaries

Definition (Crossing Number)

- P : *a planar point set in general position*
- T : *a spanning tree for* P
- **crossing number** *of* T : *maximum number of edges in* T *that can be intersected*

Fact

- P always has a spanning tree with crossing number $O(\sqrt{n})$ [Chazelle, 1989] using *iterative reweighting*
- NP-hardness of computing the optimal tree [Fekete, 2008]

Previous Work

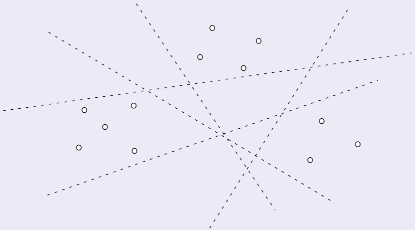
- Heuristic on iterative LP-rounding [Fekete, 2008]
- Iterative randomized rounding while solving a certain LP [Har-Peled, 2009]

Existence of Low Crossing Trees

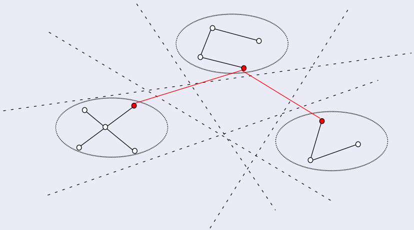
Iterative Rounding [Har-Peled, 2009]

1. Selecting edges from LP formulation
2. Forming a spanning tree

Input: P, L



Intermediate Step



LPs for the Proof

Summary

- Primal models a graph with $O(\sqrt{n})$ crossing number
- Each point $p \in P$ has an incident edge

Primal

$$\sum_{pq \in E_\ell} x_{pq} \leq \sqrt{n} \qquad \forall \ell \in L_P$$

$$\sum_{pq \in E_P} x_{pq} \geq 1 \qquad \forall p \in P$$

$$x_{pq} \geq 0 \qquad \forall pq \in E_P$$

Notation

1. E_P : set of line segments pq with $p \neq q \in P$
2. L_P : set of representative lines
3. For $\ell \in L$: $E_\ell \subseteq E_P$ set of all edges intersecting ℓ
4. For $pq \in E_P$: $L_{pq} \subseteq L_P$ set of lines intersecting pq

Notion of the Proof

Our Results

- A shorter proof
- Simplified Primal-Dual LP formulation from [Har-Peled, 2009]
- Using Farkas' Lemma

Lemma (Farkas' Lemma)

Let A be a rational $m \times n$ matrix and $b \in \mathbb{Q}^m$. **Either**

1. *there is a vector $x \in \mathbb{Q}^n$ satisfying $Ax \leq b, x \geq 0$, or*
2. *there is a vector $y \in \mathbb{Q}^m$ satisfying $A^T y \geq 0, b^T y < 0, y \geq 0$.*

Idea

- Show infeasibility of Dual LP by contradiction
- Due to Farkas' Lemma, Primal must be feasible

Overview of Algorithms

Approximation Algorithms

IterReweighting Popular framework [Welzl, 1992]

- Weighting of lines and edges
- Choosing iteratively lightest edge

Har-Peled LP Adjusted LP from proof [Har-Peld, 2009]

IterLP-rounding LP from [Fekete, 2008]

- selecting in each iteration one suitable edge
- using exponential number of constraints

A new heuristic: Connected Components Approach

- adapted from IterLP-rounding
- with polynomial many constraints

Connected Components Heuristics

Notation

- LP models only edges among the connected components $\mathcal{C}$
- $E(\mathcal{C})$ are these edges

LP

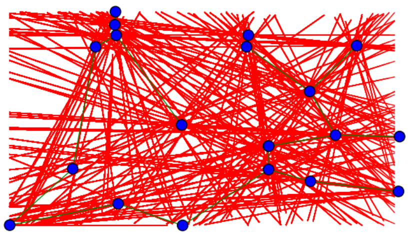
$$
\begin{aligned}
& \text{minimize } t \\
\text{s. t.} & \sum_{pq \in E(\mathcal{C})} x_{pq} = |\mathcal{C}| - 1 \\
& \sum_{pq \in E(\mathcal{C}) \colon p \in C, q \notin C} x_{pq} \geq 1 && \forall C \in \mathcal{C} \\
& \sum_{pq \in E(\mathcal{C}) \colon pq \cap \ell \neq \emptyset} x_{pq} \leq t && \forall \ell \in L \\
& x_{pq} \geq 0 && \forall pq \in E(\mathcal{C})
\end{aligned}
$$

Data Setting

P uniformly at random from integer $[n] \times [n]$ grid perturbed by an ε

All lines

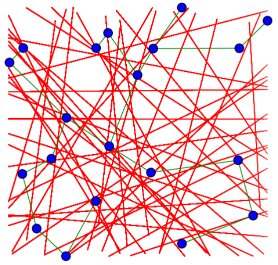
$L = L_P, |L| = \Theta(n^2)$



IterReweighting on $|P| = 20$ with all lines

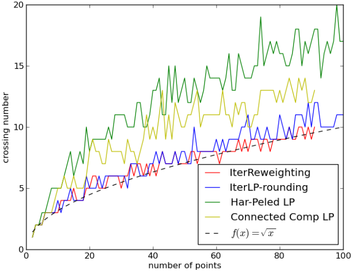
Random lines

L of size $\Theta(\sqrt{n})$

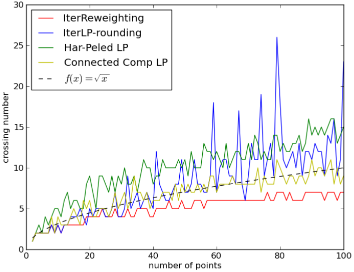


IterReweighting on $|P| = 20$ with random lines

Experimental Results on Artificial Data



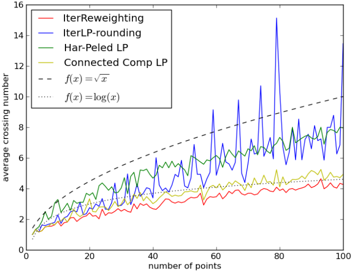
All lines



Random lines

- All algorithms produce a crossing number $O(\sqrt{n})$
- IterReweighting yields a crossing number lower than $O(\sqrt{n})$ for random lines

Average Crossing Number on Artificial Data



Average crossing number on random points with random lines

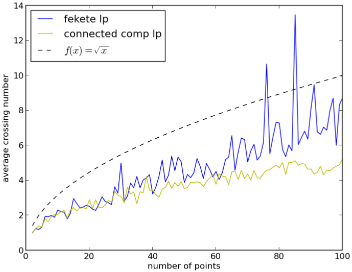
- The number of all crossings over the number of lines
- Best results by IterReweighting and Connected Components
- Yielding an average crossing number of $O(\log(n))$

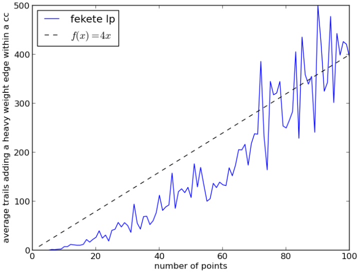
Summary

1. Alternative proof on existence of low-crossing spanning trees with crossing number $O(\sqrt{n})$
2. A new heuristic competing with existing approaches
3. Experimental results on
   - artificial data
   - real TSP instances from TSPLIB

Thank you.

Oscillation of IterLP-rounding





- Skipping of heavy weight edges might influence crossings
- Only effects IterLP-rounding
- Input data: Random points with random lines

Fekete et. al. IP

IP:

$$\text{minimize } t \tag{1}$$

$$\text{such that } \sum_{ij \in E_P} x_{ij} = n - 1 \tag{2}$$

$$\sum_{ij \in \delta(S)} x_{ij} \geq 1 \qquad \forall \emptyset \neq S \subset P \tag{3}$$

$$\sum_{ij \in E_P : ij \cap \ell \neq \emptyset} x_{ij} \leq t \qquad \forall \ell \in L \tag{4}$$

$$x_{ij} \in \{0, 1\} \qquad \forall ij \in E_P \tag{5}$$

Iterative Reweighting

Algorithm

$\textsc{IterReweighting}(G, L)$

$i \leftarrow 1$

$F \leftarrow \emptyset$

$\mathcal{C} \leftarrow \{\{1\}, \{2\}, \ldots \{n\}\}$

while $|\mathcal{C}| > 1$

do $n_{i-1}(l) \leftarrow |\{e \in F \mid e \cap \ell \neq \emptyset\}| \quad \forall \ell \in L$

$\quad w_{i-1}(l) \leftarrow 2^{n_{i-1}(l)} \quad \forall \ell \in L$

$\quad w_{i-1}(e) \leftarrow \sum_{l \in L : e \cap \ell \neq \emptyset} w_{i-1}(l) \quad \forall e \in \mathrm{E}(\mathcal{C})$

$\quad ij \leftarrow \arg\min_{pq \in \mathrm{E}(\mathcal{C})} \{w_{i-1}(pq)\}$

$\quad F \leftarrow F \cup \{ij\}$

$\quad \mathcal{C} \leftarrow \textsc{Merge}(\mathrm{C}(i), \mathrm{C}(j))$

$\quad i \leftarrow i + 1$

return F

Har-Peled's generic LP

LP for set systems with bounded VC dimension:

$$\max \sum_{pq \in E_P} y_{pq} \tag{6}$$

$$\text{such that} \sum_{pq \in E_P : |pq \cap S| = 1} y_{pq} \leq t \qquad \forall S \in \mathcal{F} \tag{7}$$

$$\sum_{q \in P : q \neq p} y_{pq} \geq 1 \qquad \forall p \in P \tag{8}$$

$$y_{pq} \geq 0 \qquad \forall pq \in E_P \tag{9}$$

What is the average crossing number?

Definition

The average crossing number for a spanning tree F is defined by:

$$\varnothing\,\mathrm{cross}(F) := \frac{1}{|L|} \sum_{\ell \in L} |\{pq \in F \mid pq \cap \ell \neq \emptyset\}|$$